Building Thinking Classrooms: Conditions for Problem Solving

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In this chapter, I first introduce the notion of a thinking classroom and then present the results of over 10 years of research done on the development and maintenance of thinking classrooms. Using a narrative style, I tell the story of how a series of failed experiences in promoting problem-solving in the classroom led first to the notion of a thinking classroom and then to a research project designed to find ways to help teachers build such a classroom. Results indicate that there are a number of relatively easy-to-implement teaching practices that can bypass the normative behaviours of almost any classroom and begin the process of developing a thinking classroom.

Motivation

My work on this paper began over 10 years ago with my research on the AHA! experience and the profound effects that these experiences have on students' beliefs and self-efficacy about mathematics (Liljedahl, 2005). That research showed that even one AHA! experience, on the heels of extended efforts at solving a problem or trying to learn some mathematics, was able to transform the way a student felt about mathematics as well as his or her ability to do mathematics. These were descriptive results. My inclination, however, was to try to find a way to make them prescriptive. The most obvious way to do this was to find a collection of problems that provided enough of a challenge that students would get stuck, and then have a solution, or solution path, appear in a flash of illumination. In hindsight, this approach was overly simplistic. Nonetheless, I implemented a number of these problems in a grade 7 (12–13 year olds) class.

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The teacher I was working with, Ms. Ahn, did the teaching and delivery of problems and I observed. Despite her best intentions the results were abysmal. The students did get stuck, but not, as I had hoped, after a prolonged effort. Instead, they gave up almost as soon as the problem was presented to them and they resisted any effort and encouragement to persist. After three days of constant struggle, Ms. Ahn and I both agreed that it was time to abandon these efforts. Wanting to better understand why our well-intentioned efforts had failed, I decided to observe Ms. Ahn teach her class using her regular style of instruction.

That the students were lacking in effort was immediately obvious, but what took time to manifest was the realization that what was missing in this classroom was that the students were not thinking. More alarming was that Ms. Ahn's teaching was predicated on an assumption that the students either could not or would not think. The classroom norms (Yackel & Rasmussen, 2002) that had been established had resulted in, what I now refer to as, a non-thinking classroom. Once I realized this, I proceeded to visit other mathematics classes—first in the same school and then in other schools. In each class, I saw the same basic behaviour—an assumption, implicit in the teaching, that the students either could not or would not think. Under such conditions, it was unreasonable to expect that students were going to spontaneously engage in problem-solving enough to get stuck and then persist through being stuck enough to have an AHA! experience.

What was missing for these students, and their teachers, was a central focus in mathematics on thinking. The realization that this was absent in so many classrooms that I visited motivated me to find a way to build, within these same classrooms, a culture of thinking, both for the student and the teachers. I wanted to build, what I now call, a *thinking classroom*—a classroom that is not only conducive to thinking but also occasions thinking, a space that is inhabited by thinking individuals as well as individuals thinking collectively, learning together and constructing knowledge and understanding through activity and discussion.

Early Efforts

A thinking classroom must have something to think about. In mathematics, the obvious choice for this is a problem-solving task. Thus, my early efforts to build thinking classrooms were oriented around problem-solving. This is a subtle departure from my earlier efforts in Ms. Ahn's classroom. Illumination-inducing tasks were, as I had learned, too ambitious a step. I needed to begin with students simply engaging in problem-solving. So, I designed and delivered a three session workshop for middle school teachers (ages 10–14) interested in bringing problem-solving into their classrooms. This was not a difficult thing to attract teachers to. At that time, there was increasing focus on problem-solving in both the curriculum and the textbooks. The research on the role of problem-solving as both an end unto itself and as a tool for learning was beginning to creep into the professional discourse of teachers in the region.

The three workshops, each 2 h long, walked teachers through three different aspects of problem-solving. The first session was focused around initiating problem-solving work in the classroom. In this session, teachers experienced a number of easy-to-start problem-solving activities that they could implement in their classrooms—problems that I knew from my own experiences were engaging to students. There were a number of mathematical card tricks to explain, some problems with dice, and a few engaging word problems. This session was called *Just do It*, and the expectation was that teachers did just that—that they brought these tasks into their classrooms and had students just do them. There was to be no assessment and no submission of student work.

The second session was called *Teaching Problem-Solving* and was designed to help teachers emerge from their students' experience a set of heuristics for problem-solving. This was a significant departure from the way teachers were used to teaching heuristics at this grade level. The district had purchased a set of resources built on the principles of Pólya's *How to Solve It* (1957). These resources were pedantic in nature, relying on the direct instruction of these heuristics, one each day, followed by some exercises for students to go through practicing the heuristic of the day. This second workshop was designed to do the opposite. The goal was to help teachers pull from the students the problem-solving strategies that they had used quite naturally in solving the set of problems they had been given since the first workshop, to give names to these strategies and to build a poster of these named strategies as a tool for future problem-solving work. This poster also formed an effective vocabulary for students to use in their group or whole class discussions as well as any mathematical writing assignments.

The third workshop was focused on leveraging the recently acquired skills towards the learning of mathematics and to begin to use problem-solving as a tool for the daily engagement in, and learning of, mathematics. This workshop involved the demonstration of how these new skills could intersect with the curriculum in general and the textbook in particular.

The series of three workshops was offered multiple times and was always well attended. Teachers who came to the first tended, for the most part, to follow through with all three sessions. From all accounts, the teachers followed through with their 'homework' and engaged their students in the activities they had experienced within the workshops. However, initial data collected from interviews and field notes were mixed. Teachers reported things like:

"Some were able to do it."
"They needed a lot of help."
"They loved it."
"They don't know how to work together."
"They got it quickly and didn't want to do anymore."
"They gave up early."

Further probing revealed that teachers who reported that their students loved what I was offering tended to have practices that already involved some level of problem-solving. If there was already a culture of thinking and problem-solving in the classroom, then this was aided by the vocabulary of the problem-solving posters, and the teachers got ideas about how to teach with problem-solving. It also revealed that those teachers who reported that their student gave up or didn't know how to work together mostly had practices devoid of problem-solving and group work. In these classrooms, although some students were able to rise to the task, the majority of the class was unable to do much with the problems—recreating, in essence, what I had seen in Ms. Ahn's class. In short, the experiences that the teachers were having implementing problem-solving in the classroom were being filtered through their already existing classroom norms (Yackel & Rasmussen, 2002).

Classroom norms are a difficult thing to bypass (Yackel & Rasmussen, 2002), even when a teacher is motivated to do so. The teachers that attended these workshops wanted to change their practice, but their initial efforts to do so were not rewarded by comparable changes in their students' problem-solving behaviour. Quite the opposite, many of the teachers I was working with were met with resistance and complaints when they tried to make changes to their practice.

From these experiences, I realized that if I wanted to build thinking classrooms—to help teachers to change their classrooms into thinking classrooms—I needed a set of tools that would allow me, and participating teachers, to bypass any existing classroom norms. These tools needed to be easy to adopt and have the ability to provide the space for students to engage in problem-solving unencumbered by their rehearsed tendencies and approaches when in their mathematics classroom.

This realization moved me to begin a program of research that would explore both the elements of thinking classrooms and the traditional elements of classroom practice that block the development and sustainability of thinking classrooms. I wanted to find a collection of teacher practices that had the ability to break students out of their classroom normative behaviour—practices that could be used not only by myself as a visiting teacher but also by the classroom teacher that had previously entrenched the classroom norms that now needed to be broken.

Thinking Classroom

As mentioned, a *thinking classroom* is a classroom that is not only conducive to thinking but also occasions thinking, a space that is inhabited by thinking individuals as well as individuals thinking collectively, learning together and constructing knowledge and understanding through activity and discussion. It is a space wherein the teacher not only fosters thinking but also expects it, both implicitly and explicitly. As such, a thinking classroom, as I conceive it, will intersect with research on mathematical thinking (Mason, Burton, & Stacey, 1982) and classroom norms (Yackel & Rasmussen, 2002). It will also intersect with notions of a didactic contract (Brousseau, 1984), the emerging understandings of studenting (Fenstermacher, 1986, 1994; Liljedahl & Allan, 2013a, 2013b), knowledge for teaching (Hill, Ball, & Schilling, 2008; Shulman, 1986) and activity theory (Engeström, Miettinen, & Punamäki, 1999).

In fact, the notion of a thinking classroom intersects with all aspects of research on teaching and learning, both within mathematics education and in general. All of these theories can be used to explain aspects of an already thinking classroom, and some of them can even be used to inform us how to begin the process of build a thinking classroom. Many of these theories have been around a long time, and yet non-thinking classrooms abound. As such, I made the decision early on to approach my work not from the perspective of a priori theory but from existing teaching practices.

General Methodology

The research to find the elements and teaching practices that foster, sustain and impede thinking classrooms has been going on for over 10 years. Using a framework of noticing (Mason, 2002),[1] I initially explored my own teaching, as well as the practices of more than 40 classroom mathematics teachers. From this emerged a set of nine elements that permeate mathematics classroom practice—elements that account for most of whether or not a classroom is a thinking or a non-thinking classroom. These nine elements of mathematics teaching became the focus of my research. They are:

1. the type of tasks used and when and how they are used
2. the way in which tasks are given to students
3. how groups are formed, both in general and when students work on tasks
4. student workspace while they work on tasks
5. room organization, both in general and when students work on tasks
6. how questions are answered when students are working on tasks
7. the ways in which hints and extensions are used, while students work on tasks
8. when and how a teacher levels[2] their classroom during or after tasks
9. assessment, both in general and when students work on tasks

Ms. Ahn's class, for example, was one in which:

1. practice tasks were given after she had done a number of worked examples
2. students either copied these from the textbook or from a question written on the board
3. students had the option to self-group to work on the homework assignment when the lesson portion of the class was done

[1] At the time, I was only informed by Mason (2002). Since then, I have been informed by an increasing body of literature on noticing (Fernandez, Llinares, & Valls, 2012; Jacobs, Lamb, & Philipp, 2010; Mason, 2011; Sherin, Jacobs, & Philipp, 2011; van Es, 2011).

[2] Levelling (Schoenfeld, 1985) is a term given to the act of closing of, or interrupting, students' work on tasks for the purposes of bringing the whole of the class (usually) up to certain level of understanding. It is most commonly seen when a teacher ends students work on a task by showing how to solve the task.

4. students worked at their desks, writing in their notebooks
5. students sat in rows with the students' desk facing the board at the front of the classroom
6. students who struggled were helped individually through the solution process, either part way or all the way
7. there were no hints, only answers, and an extension was merely the next practice question on the list
8. when 'enough time' time had passed, Ms. Ahn would demonstrate the solution on the board, sometimes calling on 'the class' to tell her how to proceed
9. assessment was always through individual quizzes and tests

This was not, as determined earlier, a thinking classroom. Each of these elements was something that needed exploring and experimenting with. Many were steeped in tradition and classroom norms (Yackel & Rasmussen, 2002).

Research into each of these was done using design-based methods (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Design-Based Research Collective, 2003)[3] within both my own teaching practice as well as the practices of a number of teachers participating in a variety of professional development opportunities. This approach allowed me to vary the teaching around each of the elements, either independently or jointly, and to measure the effectiveness of that method for building and/or maintaining a thinking classroom. Results fed recursively back into teaching practice, each time leading either to refining or abandoning what was done in the previous iteration.

This method, although fruitful in the end, presented two challenges. The first had to do with the measurement of effectiveness. To do this, I used what I came to call *proxies for engagement*—observable and measurable (either qualitatively or quantitatively) student behaviours. At first, this included only behaviours that fit the *a priori* definition of a thinking classroom. As the research progressed, however, the list of these proxies grew and changed depending on the element being studied and teaching method being used.

The second challenge had to do with the shift in practice needed when it was determined that a particular teaching method needed to be abandoned. Early results indicated that small shifts in practice did little to shift the behaviours of the class as a whole. Larger, more substantial shifts were needed. These were sometimes difficult to conceptualize. In the end, a contrarian approach was adopted. That is, when a teaching method around a specific element needed to be abandoned, the new approach to be adopted was, as much as possible, the exact opposite to the practice that had shown to be ineffective for building or maintaining a thinking classroom. When sitting showed to be ineffective, we tried making the students stand. When levelling to the top failed, we tried levelling to the bottom. When answering questions proved to be ineffective, we stopped answering questions. Each of these

[3] This research is now informed also by Norton and McCloskey (2008) and Anderson and Shattuck (2012).

approaches needed further refinement through the iterative design-based research approach, but it gave good starting points for this process.

In what follows, I will first present the results of the research done on two of these elements—student workspace and how groups are formed—both independently and jointly. I then present, in brief, the results of the research done on the remaining seven elements and discuss how all nine elements hold together as a framework to build and maintain thinking classrooms. All of this research is informed dually by data and analysis that looks both on the effect on students and the uptake by teachers.

Student Workspace

The research on student workspace began by looking at the default—students sitting in their desks. It became obvious early in this work that this was not conducive to the building of a thinking classroom. As such, almost immediately, a new space was explored. Following the contrarian approach established early on, the next space to test was to have students standing and working somewhere other than at their desks. The shift to having students work on whiteboards and blackboards was then an obvious extension.

In many classrooms where the research was being done, however, there were not enough whiteboards and blackboards available for all groups to work at. Some students would have to still be seated in their desks. This led to a phase of experimentation with alternative work surfaces, including poster board or flipchart paper attached to the walls and smaller whiteboards laying on desks—with some classrooms using all three at the same time. Whenever this occurred, there was a general sense shared between whatever teachers were in the room, as well as myself, that the vertical whiteboards were superior to any of the other options available to students. These observations led to the following pseudo-quantitative study focusing on this phenomenon.

Participants

The participants for this study were the students in five high school classrooms; two grade 12 ($n=31$, 30), two grade 11 ($n=32$, 31) and one grade 10 ($n=31$).[4] In each of these classes, students were put into groups of two to four and assigned to one of five work surfaces to work on while solving a given problem-solving task.

[4] In Canada, grade 12 students are typically 16–18 years of age, grade 11 students 15–18 and grade 10 students 14–17. The age variance is due to a combination of some students fast-tracking to be a year ahead of their peers and some students repeating or delaying their grade 11 mathematics course.

Participating in this phase of the research were also the five teachers whose classes the research took place in. Most high school mathematics teachers teach anywhere from three to seven different classes. As such, it would have been possible to have gathered all of the data from the classes of a single teacher. In order to diversify the data, however, it was decided that data would be gathered from classes belonging to five different teachers.

These teachers were all participating in one of several learning teams which ran in the fall of 2006 and the spring of 2007. Teachers participated in these teams voluntarily with the hope of improving their practice and their students' level of engagement. Each of these learning teams consisted of between 4 and 6, a 2-h meeting spread over half a school year. Sessions took teachers through a series of activities modelled on my most current knowledge about building and maintaining thinking classrooms. Teachers were asked to implement the activities and teaching methods in their own classrooms between meetings and report back to the team how it went.

The teachers, whose classrooms this data was collected in, were all new to the ideas being presented and, other than having individual students occasionally demonstrate work on the whiteboard at the front of the room, had never used them for whole class activity.

Data

As mentioned, the students, in groups of 2–4, worked on one of five assigned work surfaces: a wall-mounted whiteboard, a whiteboard laying on top of their desks or table, a sheet of flipchart paper taped to the wall, a sheet of flipchart paper laying on top of their desk or table, and their own notebooks at their desks or table. To increase the likelihood that they would work as a group, each group was provided with only one felt or, in the case of working in a notebook, one pen. To measure the effectiveness of each of these surfaces, a series of *proxies for engagement* were established.

It is not possible to measure how much a student is thinking during any activity, or how that thinking is individual or predicated on and with the other members of his or her group. However, there are a variety of proxies for this level of engagement that can be established—*proxies for engagement*. For the research presented here, a variety of objective and subjective proxies were established.

1. *Time to task*
   This was an objective measure of how much time passed between the task being given and the first discernable discussion as a group about the task.
2. *Time to first mathematical notation*
   This was an objective measure of how much time passed between the task being given and the first mathematical notation was made on the work surface.
3. *Eagerness to start*
   This is a subjective measure of how eager a group was to start working on a task. A score of 0, 1, 2 or 3 was assigned with 0 being assigned for no enthusiasm

to begin and a 3 being assigned if every member of the group were wanting to start.

4. *Discussion*
   This is a subjective measure of how much group discussion there was while working on a task. A score of 0, 1, 2 or 3 was assigned with 0 being assigned for no discussion and a 3 being assigned for lots of discussion involving all members of the group.
5. *Participation*
   This is a subjective measure of how much participation there was from the group members while working on a task. A score of 0, 1, 2 or 3 was assigned with 0 being assigned if no members of the group were active in working on the task and a 3 being assigned if all members of the group were participating in the work.
6. *Persistence*
   This is a subjective measure of how persistent a group was while working on a task. A score of 0, 1, 2 or 3 was assigned with 0 being assigned if the group gave up immediately when a challenge was encountered and a 3 being assigned if the group persisted through multiple challenges.
7. *Non-linearity of work*
   This is a subjective measure of how non-linear groups work was. A score of 0, 1, 2 or 3 was assigned with 0 being assigned if the work was orderly and linear and a 3 being assigned if the work was scattered.
8. *Knowledge mobility*
   This is a subjective measure of how much interaction there was between groups. A score of 0, 1, 2 or 3 was assigned with 0 being assigned if there was no interaction with another group and a 3 being assigned if there were lots of interaction with another group or with many other groups.

These measures, like all measures, are value laden. Some proxies (1, 2, 3, 6) were selected partially from what was observed informally when being in a setting where multiple work surfaces were being utilized. Others proxies (4, 5, 7, 8) were selected specifically because they embody some of what defines a thinking classroom—discussion, participation, non-linear work, and knowledge mobility.

As mentioned, these data were collected in the five aforementioned classes during a group problem-solving activity. Each class was working on a different task. Across the five classes, there were ten groups that worked on a wall-mounted whiteboard, ten that worked on a whiteboard laying on top of their desks or table, nine that worked on flipchart paper taped to the wall, nine that worked on flipchart paper laying on top of their desk or table, and eight that worked in their own notebooks at their desks or table. For each group, the aforementioned measures were collected by a team of 3–5 people: the teacher whose class it was, the researcher (me), as well a number of observing teachers. The data were recorded on a visual representation of the classroom and where the groups were located with no group being measured by more than one person.

Results and Discussion

For the purposes of this chapter, it is sufficient to show only the average scores of this analysis (see Table 1).

The data confirmed the informal observations. Groups are more eager to start and there is more discussion, participation, persistence and non-linearity when they work on the whiteboards. However, there are nuances that deserve further attention. First, although there is no significant difference in the time it takes for the groups to start discussing the problem, there is a big difference between whiteboards and flipchart paper in the time it takes before groups make their first mathematical notation. This is equally true whether groups are standing or sitting. This can be attributed to the non-permanent nature of the whiteboards. With the ease of erasing available to them, students risk more and risk sooner. The contrast to this is the very permanent nature of a felt pen on flipchart paper. For students working on these surfaces, it took a very long time and much discussion before they were willing to risk writing anything down. The notebooks are a familiar surface to students, so this can be discounted with respect to willingness to risk starting.

Although the measures for the whiteboards are far superior to that of the flipchart paper and notebook for the measures of eagerness to start, discussion, and participation, it is worth noting that in each of these cases, the vertical surface scores higher than the horizontal one. Given that the maximum score for any of these measures is 3, it is also worth noting that eagerness scored a perfect 3 for those that were standing. That is, for all ten cases of groups working at a vertical whiteboard, ten independent evaluators gave each of these groups the maximum score. For discussion and participation, eight out of the ten groups received the maximum score. On the same measures, the horizontal whiteboard groups received 3, 3, and 2 maximum scores, respectively. This can be attributed to the fact that sitting, even while working at a whiteboard, still gives students the opportunity to become anonymous, to hide and to not participate. Standing doesn't afford this.

Table 1 Average times and scores on the eight measures

	Vertical whiteboard	Horizontal whiteboard	Vertical paper	Horizontal paper	Notebook
N (groups)	10	10	9	9	8
1. Time to task	12.8 s	13.2 s	12.1 s	14.1 s	13.0 s
2. Time to first notation	20.3 s	23.5 s	2.4 min	2.1 min	18.2 s
3. Eagerness	3.0	2.3	1.2	1.0	0.9
4. Discussion	2.8	2.2	1.5	1.1	0.6
5. Participation	2.8	2.1	1.8	1.6	0.9
6. Persistence	2.6	2.6	1.8	1.9	1.9
7. Non-linearity	2.7	2.9	1.0	1.1	0.8
8. Mobility	2.5	1.2	2.0	1.3	1.2

With respect to non-linearity, it is clear that the whiteboards, either vertical or horizontal, allow a greater freedom to explore the problem across the entirety of the surface. Although the whiteboards provide an ease of erasing that is not afforded on the flipchart, work is rarely erased by the students working on whiteboard surfaces. It seems that rather than erasing to make room for more work, the workspace migrates around the whiteboard surface, representing the chronological nature of problem-solving. In contrast, the groups working on flipchart paper tended to not write any work down until they were clear it would contribute to the logical development of a solution.

Finally, it is worth noting that groups that were standing also were more likely to engage with other groups that were standing close by. Although not measured, it was clear that this was more true for the vertical whiteboard groups. There are a number of reasons for this. Most obvious, vertical surfaces are more visible. However, there were very few observed instances of groups that were sitting down looking up to see what the groups that were standing were doing. Likewise, there were no instances of the students standing, looking at the work of the groups that were sitting. Amongst those that were standing, there was a lot of interaction between those working on whiteboards, and almost none between those working on flipchart paper. Finally, there was very little interaction between those working on flipchart paper and those working on whiteboards. Part of this can be explained by proximity—the whiteboard groups were clustered on one or two whiteboards, while the flipchart people were clustered elsewhere. But it also is the case that the whiteboard groups had little reason to look to the flipchart groups. They worked slower and had little written on their work surfaces. This was also true between the flipchart groups—there was little to look at.

In short, groups that worked on vertical whiteboards demonstrated more thinking classroom behaviour—persistence, discussion, participation and knowledge mobility—than any of the other types of work surfaces. The next most conducive was a horizontal whiteboard. The remaining three were not only not conducive to promoting thinking classroom behaviour but they may actually have inhibited it. From this it is clear that the non-permanence of surfaces is critical for decreasing time to task, as well as improving enthusiasm, discussion, participation, and persistence. It also increases the non-linearity of work which mirrors the actual work of thinking groups. Making these non-permanent surfaces vertical further enhances all of these qualities, as well as fostering inter-group collaboration, something that is needed to move the class from a collection of thinking groups to being a thinking classroom.

Vertical Non-permanent Surfaces: Teacher Uptake

Having this evidence that vertical non-permanent surfaces (VNPS) are so instrumental in the fostering of thinking classroom behaviour, a follow-up study was done with teachers vis-à-vis the use of this work surface. The goal of this follow-up

Table 2 Distribution of participants in VNPS study

	Elementary	Middle	Secondary	Total
Learning team	21	43	41	105
Multi-session workshops	12	28	42	82
Single workshops	35	24	54	113
Total	68	95	137	**300**

study was to see the degree to which teachers, when presented with the idea of non-permanent vertical surfaces, were keen to implement it within their teaching, actually tried it, and continued to use it in their teaching.

Participants

Participants for this portion of the study were 300 in-service teachers of mathematics—elementary, middle and secondary school. They were drawn from three sources over a four-year period (2007–2011): participants in variety of single workshops, participants in a number of multi-session workshops, and participants in learning teams. The breakdown of participants, according to grade levels, and form of contact is represented in Table 2.

There were a number of teachers who attended a combination of learning teams, multi-session workshops and single workshops. In these cases, their data was registered as belonging to the group with the most contact. That is, if they attended a single workshop, as well as being a member of a learning team, their participation was registered as being a member of a learning team.

These participants are only a subset of all the teachers that participated in these learning teams, multi-session workshops, and single workshops. They were selected at random from each group I worked with by approaching them at the end of the first (and sometimes only) session and asking them if they would be willing to have me contact them and, potentially, visit their classrooms.

Data

Data consists primarily of interview data. Each participant was interviewed immediately after a session where they were first introduced to the idea of vertical non-permanent surfaces, 1 week later, and 6 weeks later. These interviews were brief and, depending on when the interview was conducted, was originally designed to gauge the degree to which they were committed to trying, or continuing to use, vertical non-permanent surfaces in their teaching and how they were using them. However, participants wanted to talk about much more than just this. They wanted to discuss innovations they had made, the ways in which this was changing their teaching practice as a whole, the reactions of the students and their colleagues, as well as a variety of other details pertaining to vertical non-permanent surfaces. With time, these impromptu conversations changed the initial interview questions to begin to also probe for these more nuanced details. For the purposes of this chapter, however, only the data pertaining to the original intent will be presented.

In addition to the interview data, there were also field notes from 20 classroom visits. These visits were implemented for the purposes of checking the fidelity of the interview data—to see if what teachers were saying is actually what they were doing. In each case, this proved to be the case. It was clear from these data that teachers were true to their words with respect to their use of vertical non-permanent surfaces. However, these visits, like the interviews, offered much more than what was expected. I saw innovations in implementation, observed the enthusiasm of the students, and witnessed the transformational effect that this was having on the teaching practices of the participants.

Results and Discussion

In general, almost all of the teachers who were introduced to the notion of vertical non-permanent surfaces were determined to try it within their teaching and were committed to keep doing it, even after 6 weeks (see Fig. 1). This is a significant uptake rarely seen in the literature. This is likely due, in part, to the ease with which it is modelled in the various professional development settings. During these sessions, not only are the methods involved easily demonstrated but the teachers immediately feel the impact on themselves as learners when they are put into a group to work on a vertical non-permanent surface.

An interesting result from this aggregated view is that there were more teachers using non-permanent vertical surfaces after 6 weeks than there was after 1 week. This has to do with access to these vertical non-permanent surfaces. Many teachers struggled to find such surfaces. There were some amazing improvisations in this regard, from using windows to bringing in a number of novel surfaces, from shower curtains to glossy wall boards. One teacher even stood her classroom tables on end to achieve the effect. As time went on, teachers were able to convince their administrators to provide them with enough whiteboards that these improvisations no longer became necessary. For some teachers, this took more time than others and speaks to the delayed uptake seen in Fig. 1. However, it also speaks to the persistence with which many teachers pursued this idea with.

A disaggregated look at the data shows that neither the grade levels being taught (see Fig. 2) or the type of professional development setting in which the idea was presented (see Fig. 3) had any significant impact on the uptake.

Literature on teacher change typically implies that sustained change can only be achieved through professional development opportunities with multiple sessions and extended contact. That is, single workshops are not effective mediums for promoting change (Jasper & Taube, 2004; Little & Horn, 2007; Lord, 1994; McClain

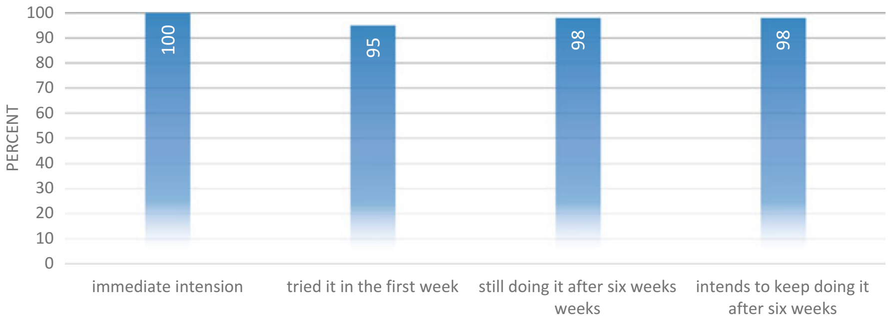


Fig. 1 Uptake of VNPS ($n=300$)

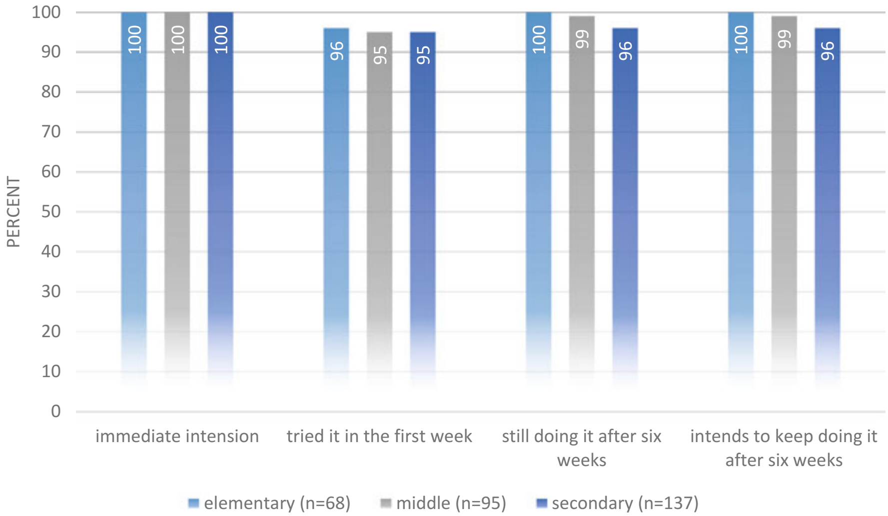


Fig. 2 Uptake of VNPS by grade levels ($n=300$)

& Cobb, 2004; Middleton, Sawada, Judson, Bloom, & Turley, 2002; Stigler & Hiebert, 1999; Wenger, 1998). The introduction of vertical non-permanent surfaces as a workspace doesn't adhere to these claims. There are many possible reasons for this. The first is that the introduction of non-permanent vertical surfaces was achieved in a single workshop could be, as mentioned, due to the simple fact that it is a relatively easy idea for a workshop leader to model and for workshop participants to experience. Forty five minutes of solving problems in groups standing at a whiteboard coupled with a whole group discussion on the affordances of recreating this within their own classrooms is enough to convince teachers to try it. And trying it leads to a successful implementation. Unlike many other changes that can be made in a teacher's practice, vertical non-permanent surfaces (as demonstrated in

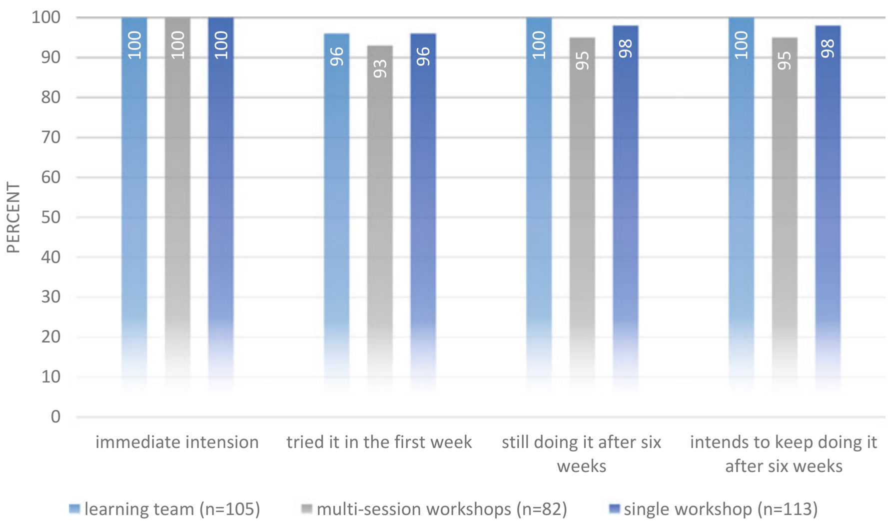


Fig. 3 Uptake of VNPS by professional development setting ($n = 300$)

the first study) was well received by students, was easy to manage at a whole class level, and had an immediate positive effects on classroom thinking behaviour. Together, the ease of modelling coupled with a successful implementation meant that vertical non-permanent surfaces did not need more than a single workshop to change teaching practice.

These possible reasons are supported by the comments of teachers from the interviews after week 1 and week 6. The following comments were chosen from the many collected for their conciseness.

> "*I will never go back to just having students work in their desks*."
> "*How do I get more whiteboards*?"
> "*The principal came into my class … now I'm doing a session for the whole staff on Monday*."
> "*My grade-partner is even starting to do it*."
> "*The kids love it. Especially the windows*."
> "*I had one girl come up and ask when it will be her turn on the windows*."

Not only is the implementation of vertical non-permanent surfaces immediately effective for these teachers, it is also infectious with other teachers quickly latching on to it and administrators quickly seeing the affordances it offers.

But if vertical non-permanent surfaces are the solution, what was the problem? When I began the research on students' workspace, the default was students sitting in desks—sometimes individually in rows, other times clustered in groups. The move from the desks to the vertical workspaces was made, not because I saw something specifically wrong with students being in desks, but rather through adherence to the contrarian approach that was adopted early on in the more general research project. Looking back now at students working in desks, from the perspective of the affordances that having them stand at a non-permanent vertical surface offers, I see more clearly the problems that desks introduced into my efforts to build and maintain thinking classrooms. Primarily, this has to do with anonymity and how desks allow for and even promote this. When students stand at a whiteboard or a window, they are all visible. There is nowhere to hide. When students are in their desks, it is easy for them to become anonymous, hidden and safe—from participating and from contributing. It is not that all students want to be hidden, to not participate, but when the problems gets difficult, when the discussions require more thinking, it is easy for a student to pull back in their participation when they are sitting. Standing in a group makes this more difficult. Not only is it immediately visible to the teacher but it is also clear to the student who is pulling back. To pull back means to step towards the centre of the room, towards the teacher, towards nothing. There is no anonymity in this.

Forming Groups

The research into how best to form groups began, like it did with student work surfaces, by looking at how groups are typically formed in a classroom. In most cases, this is either a strategically planned arrangement decided by the teacher or self-selected groupings of friends as decided by the students. Teachers tend to make groupings in order to meet their educational goals. These may include goals around pedagogy, student productivity, or simply the construction of a peaceful work environment. Meanwhile, students, when given the opportunity, tend to group themselves according to their social goals. This mismatch between educational and social goals in classrooms creates conditions where, no matter how strategic a teacher is in her groupings, some students are unhappy in the failure of that grouping to meet their social goals (Kotsopoulos, 2007; Slavin, 1996).

This disparity results in a decrease in the effectiveness of group work. This led to the exploration of alternative grouping methods. The fact that strategic grouping strategies were often not working, coupled with the contrarian approach of action in such instances, meant that random grouping methods needed to be explored. Working with the same type of population of teachers described above, a variety of random grouping methods were implemented and studied. This preliminary research showed, very quickly, that there was little difference in the effectiveness of strategic groupings and randomized groupings when the randomization was done out of sight of the students. The students assumed that all groupings had a hidden agenda, and merely saying that they were randomly generated was not enough to change classroom behaviour.

However, when the randomization was done in full view of the students, changes were immediately noticed. When randomization was done frequently—twice a day in elementary classrooms and every class in middle and secondary classrooms—the changes in classroom behaviour was profound. Within 2–3 weeks:

- Students became agreeable to work in any group they were placed in.
- There was an elimination of social barriers within the classroom.

- Mobility of knowledge between students increased.
- Reliance on the teacher for answers decreased.
- Reliance on co-constructed intra- and inter-group answers increased.
- Engagement in classroom tasks increased.
- Students became more enthusiastic about mathematics class.

To confirm these observations, one grade 10 (age 15–16) was studied. The details and results of this research have already been published in a chapter entitled *The Affordances of Using Visibly Random Groups in a Mathematics Classroom* (Liljedahl, 2014). What follows is a summary of this research.

The class in which the study was done belonged to Ms. Carley, a teacher with eight years experience who was a participant in one of the learning teams I was leading. Ms. Carley had joined the team because she was dissatisfied with the results of group work in her teaching. She knew that group work was important to learning, but, until now, had felt that her efforts in this regard had been unsuccessful. She was looking for a better way. So, when I suggested to the group that they try using visibly random groups she made an immediate commitment to start using this method in one of her classrooms.

Data consisted of interview transcripts and field notes collected over a 3-month period immediately prior to and during an implementation of visibly random groups in Ms. Carley's class. These data were analysed using analytic induction (Patton, 2002) anchored in the *a priori* and grounded observations from my initial experimentation with random groupings.

These results both confirmed and nuanced the initial observations. Students very quickly shed their anxieties about what groups they were in. They began to collaborate in earnest. After three weeks, a *porosity* developed between group boundaries as both intra- and inter-group collaboration flourished. With this heightened mobilization of knowledge came a decrease in the reliance on the teacher as the *knower* in the room. In the end, there was a marked heightening of enthusiasm and engagement for problem-solving in particular, and in mathematics class in general. In short, Ms. Carley's class became a thinking classroom.

Visibly Random Groupings: Teacher Uptake

Similar to the research on the vertical non-permanent surfaces a pseudo-quantitative study was done on the uptake by teachers on the idea of visibly random groupings (VRG). Tapping into the similar populations of teachers engaged in learning teams, multi-session workshops, and single workshops between 2009 and 2011, a population of 200 teachers were selected to participate (see Table 3).

These teachers were introduced to the idea of visibly random groupings in a similar fashion as above—through modelling and immersion. They were likewise interviewed immediately after their professional development experience, 1 week after their experience, and 6 weeks after. The results of this analysis can be seen in Fig. 4.

Table 3 Distribution of participants in VRG study

	Elementary	Middle	Secondary	Total
Learning team	15	22	31	68
Multi-session workshops	25	19	14	58
Single workshops	10	25	39	74
Total	50	66	84	**200**

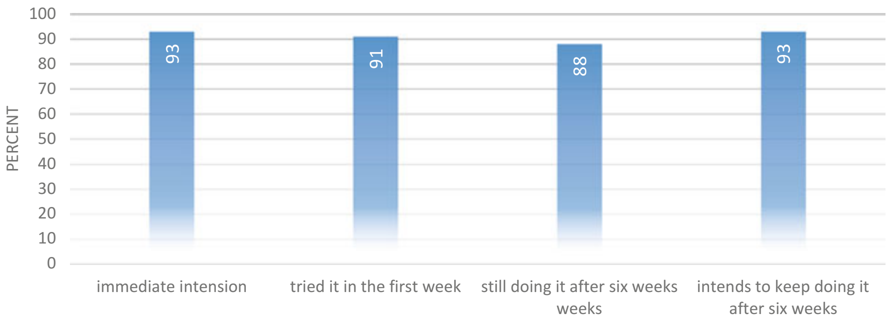


Fig. 4 Uptake of VRG ($n = 200$)

The dip in the uptake between week 1 and week 6 was minor. What was interesting was the uptick in intension after week 6. From the interviews, it became clear that the teachers who had come away from using visibly random groups did so because, after 3–4 weeks, things were working so well that they thought they could now allow the students to work with who they wanted. Once they saw that this was not as effective, they recommitted to going back to random groupings.

Like with vertical non-permanent surfaces, there was no discernible difference in uptake between elementary, middle or secondary teachers. However, unlike the previous study, there was a slight difference depending on the nature of the professional development environment they were participating in (see Fig. 5).

From the interviews, it seemed that although the immediate delivery of the idea was accomplished within a single session, the support of the learning team helped teachers to get on board late if they hesitated in implementing in the 1st week. This explains the uptick in the number of learning team members who started using randomized groups in between the first and the sixth week. This also explains why there was no such uptick amongst the single workshop participants who had no follow-up session, or amongst the multi-session participants who did not have a second session until 8 weeks after the initial idea was presented.

Regardless, there was still a significant uptake by those teachers who only experienced one 90 min session on the use of visibly random groupings. This can be explained in the same way as it was for the vertical non-permanent surfaces—it was easily modelled and the affordances became immediately apparent. As well, the students took to it quickly with little resistance once the participants implemented it within their own classrooms.

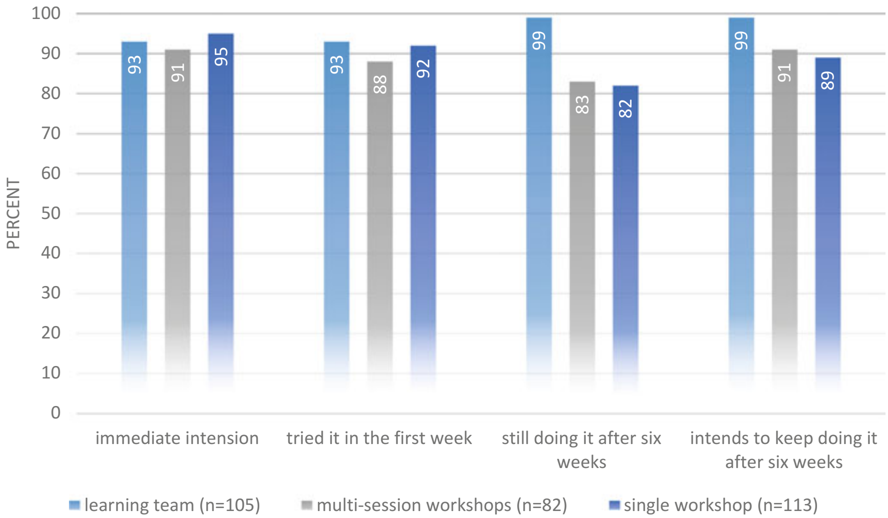


Fig. 5 Uptake of VRG by professional development setting ($n = 200$)

As with the research on the vertical non-permanent surfaces, the research on visibly random groupings included 14 classroom visits. Unlike the research on VNPS, however, the purpose of these visits was not to check the fidelity of the interview data. Rather, it was to see if teachers were continuing to use VRG's even 6–9 months after their last work with me. In each of the 14 visits, I saw a continued use of VRG strategies. And like with my visits in the VNPS research, these visits offered much more than what was expected. I saw innovations in implementation, observed the enthusiasm of the students, and witnessed the transformational effect that this was having on teaching practices.

VNPS and VRG Taken Together: Teacher Uptake

Once it was established that both vertical non-permanent surfaces and visibly random groupings were effective practices for building aspects of a thinking classroom and that these methods had good uptake by teachers, it was easy to bring them together. From a professional development perspective, this is no more difficult than presenting each one separately. VNPS and VRG are easily modelled together, with the participants being put into visibly random groupings to work on vertical non-permanent surfaces. So, this is what was done with a population of teachers similar to the ones described above. From this, 124 participants were followed to gauge the uptake of being exposed to both of these methods simultaneously. The results can be seen in Fig. 6.

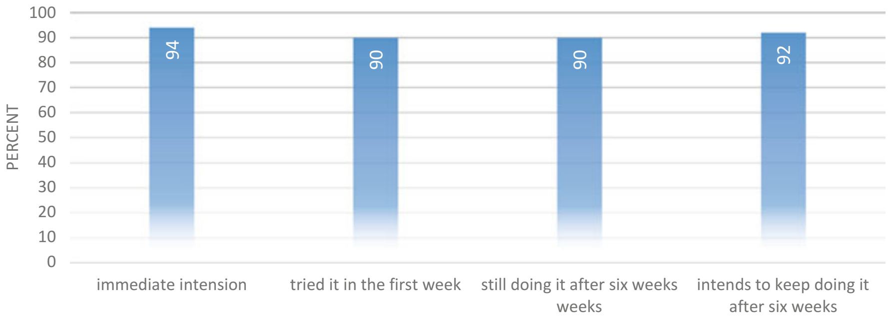


Fig. 6 Uptake of both VNPS and VRG ($n = 124$)

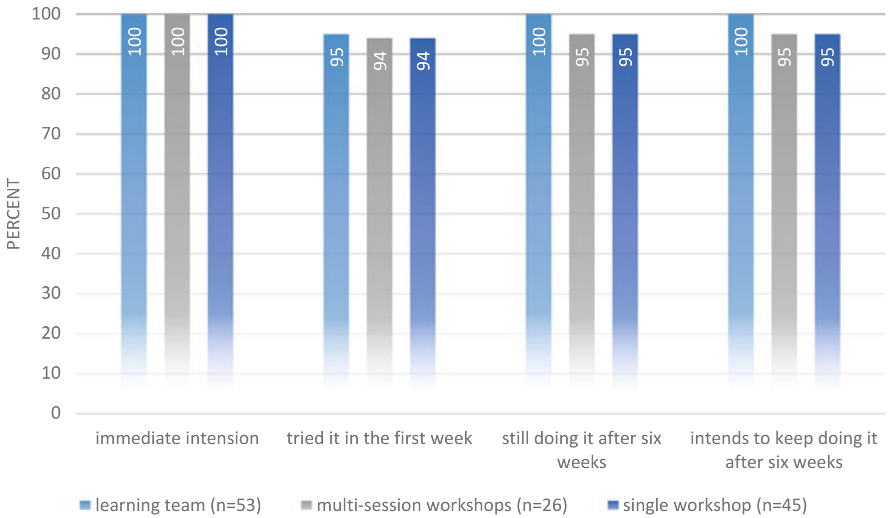


Fig. 7 Uptake of VNPS and VRG by professional development setting ($n = 124$)

Like with visibly random groupings, there was no significant difference in uptake by grade level and a slight difference in uptake as disaggregated by the professional development setting in which the combined methods were presented. Like with visibly random groupings, the teachers in the learning team setting were more consistently implementing the methods presented, whereas those teachers in the single workshop sessions were less likely to get on board late and more likely to drop off early (see Fig. 7). Despite these differences, however, the uptake across for each group was impressive with much enthusiasm for it.

With respect to the effect on students, my observations during ten classroom visits showed the combined benefits of the two interventions. The fact that the students were so comfortable working with each other, coupled with the high visibility of the work afforded by the vertical surfaces, allowed for enhanced intra-group knowledge mobilization. The teachers often commented that they saw huge improvements in the classroom community.

> *"I used to think I had a community in my classroom. Now I see what a community can look like."*

My observation of the student actions during these ten classroom visits confirmed this.

General Findings: All Nine Elements

The results from research on students' workspace and grouping methods are indicative of the findings of research into each of the nine aforementioned elements. From the design-based research on each of these—independently or in conjunction with others—emerged a set of teaching practices that are conducive to either the building, or maintenance, of a thinking classroom. In what follows briefly, these are:

1. *The type of tasks used and when and how they are used*
   Lessons need to begin with good problem-solving tasks. At the early stages of building a thinking classroom, these tasks need to be highly engaging, collaborative tasks that drive students to want to talk with each other as they try to solve them (Liljedahl, 2008). Once a thinking classroom is established, the problems need to permeate the entirety of the lesson and emerge rich mathematics (Schoenfeld, 1985) that can be linked to the curriculum content to be 'taught' that day.
2. *The way in which tasks are given to students*
   Tasks need to be given orally. If there are data or diagrams needed, these can be provided on paper, but the instructions pertaining to the activity of the task need to be given orally. This very quickly drives the groups to discuss what is being asked rather than trying to decode instructions on a page.
3. *How groups are formed, both in general and when students work on tasks*
   As presented above, groupings need to be frequent and visibly random. Ideally, at the beginning of every class, a visibly random method is used to assign students to a group of 2–4 for the duration of that class. These groups will work together on any assigned problem-solving tasks, sit together or stand together during any group or whole class discussions.
4. *Student workspace while they work on tasks*
   As discussed, groups of students need to work on vertical non-permanent surfaces such as whiteboards, blackboards, or windows. This will make visible all work being done, not just to the teacher but to the groups doing the work. To facilitate discussion, there should be only one felt pen or piece of chalk per group.
5. *Room organization, both in general and when students work on tasks*
   The classroom needs to be de-fronted. The teacher must let go of one wall of the classroom as being the designated teaching space that all desks are oriented

towards. The teacher needs to address the class from a variety of locations within the room and, as much as possible, use all four walls of the classroom. It is best if desks are placed in a random configuration around the room.

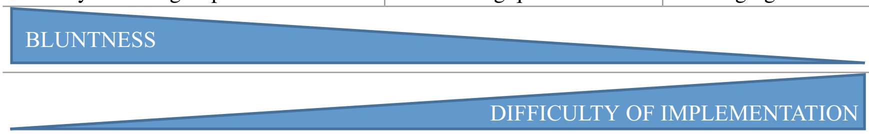
6. *How questions are answered when students are working on tasks*
   Students only ask three types of questions: (1) proximity questions—asked when the teacher is close; (2) stop-thinking questions—most often of the form 'is this right'; and (3) keep-thinking questions—questions that students ask so they can get back to work. Only the third of these types should be answered. The first two types need to be acknowledged but not answered.
7. *The ways in which hints and extensions are used while students work on tasks*
   Once a thinking classroom is established, it needs to be nurtured. This is done primarily through how hints and extensions are given to groups as they work on tasks. Flow (Csíkszentmihályi 1990, 1996) is a good framework for thinking about this. Hints and extensions need to be given so as to keep students in a perfect balance between the challenge of the current task and their abilities in working on it. If their ability is too high, the risk is they get bored. If the challenge is too great, the risk is they become frustrated.
8. *When and how a teacher levels their classroom during or after tasks*
   Levelling needs be done at the bottom. When every group has passed a minimum threshold, the teacher needs to engage in discussion about the experience and understanding the whole class now shares. This should involve a reification and formalization of the work done by the groups and often constitutes the 'lesson' for that particular class.
9. *Assessment, both in general and when students work on tasks*
   Assessment in a thinking classroom needs to be mostly about the involvement of students in the learning process through efforts to communicate with them where they are and where they are going in their learning. It needs to honour the activities of a thinking classroom through a focus on the processes of learning more so than the products and it needs to include both group work and individual work.

Discussion

However, this research also showed that these are not all equally impactful or purposeful in the building and maintenance of a thinking classroom. Some of these are blunt instruments capable of leveraging significant changes while others are more refined, used for the fine-tuning and maintenance of a thinking classroom. Some are necessary precursors to others. Some are easier to implement by teachers than others, while others are more nuanced, requiring great attention and more practice as a teacher. And some are better received by students than others. From the whole of these results emerged a three-tier hierarchy that represent not only the bluntness and ease of implementation but also an ideal chronology of implementation (see Table 4).

Table 4 Nine elements as chronologically implemented

Stage one	Stage two	Stage three
• Begin lessons with problem-solving tasks	• Oral instructions	• Levelling
• Vertical non-permanent surfaces	• De-fronting the room	• Assessment
• Visibly random groups	• Answering questions	• Managing flow



In the aforementioned research, I presented the results of research into teachers implementing teaching practices from stage one, either separately or together. However, the effect on these teachers is more profound than the numbers and graphs indicated above. This experience with elements in stage one propels them to thirst for more, both in particular and in general. They want more tasks, more examples of how to make random groupings, how to find vertical surfaces. But they also want to know more about assessment, how to ask and answer questions, how to organize their rooms, how to give instructions and how to sustain the engagement they have experienced while at the same time feeling like they are getting through the curriculum. In short, their experience with the teaching methods associated with stage one elements is quite naturally propelling them into wanting to engage in the elements in stages two and three.

These results are not definitive, exhaustive or unique. The teaching methods that emerged as effective for each of these elements emerged as a result of an *a priori* commitment to make change in a contrarian fashion. This continued until positive effects began to emerge, at which point refinements were recursively explored. It is possible that a different approach to the research would have yielded different methods. Different methods could, likewise, emerge a different set of stages optimal for the development of thinking classrooms.

Conclusions

The main goal of this research is about finding ways to build thinking classrooms. One of the sub-goals of this work on building thinking classrooms was to develop methods that not only fostered thinking and collaboration but also bypassed any classroom norms that would potentially inhibit this from happening. Using the methods in stage one while solving problems, either together or separately, was almost universally successful. They worked for any grade, in any class and for any teacher. As such, it can be said that these methods succeeded in bypassing whatever norms existed in the over 600 classrooms in which these methods were tried. Further, they not only bypassed the norms for the students but also the norms of the teachers implementing them. So different were these methods from the existing practices of the teachers participating in the research that they were left with what I have come to call *first-person vicarious experiences*. They are first person because they are living the lesson and observing the results created by their own hands. But the methods are not their own. There has been no time to assimilate them into their own repertoire of practice or into the schema of how they construct meaningful practice. They simply experienced the methods as learners and then were asked to immediately implement them as teachers. As such, they experienced a different way in which their classroom could look and how their students could behave. They experienced, through these other*ly* methods, an other*ly* classroom—a thinking classroom.

The results of this research sound extraordinary. In many ways, they are. It would be tempting to try to attribute these to some special quality of the professional development setting or skill of the facilitator. But these are not the source of these remarkable results. The results, I believe, lie not in what is new but what is not old. The classroom norms that permeate classrooms in North America, and around the world, are so robust, so entrenched, that they transcend the particular classrooms and have become institutional norms (Liu & Liljedahl, 2012). What the methods presented here offer is a violent break from these institutional norms, and in so doing, offer students a chance to be learners much more so than students (Liljedahl & Allan, 2013a, 2013b).

By constructing a thinking classroom, problem-solving becomes not only a means but also an end. A thinking classroom is shot through with rich problems. Implementation of each of the aforementioned methods associated with the nine elements and three stages relies on the ubiquitous use of problem-solving. But at the same time, it also creates a classroom conducive to the collaborative solving of problems.

Afterword

Since this research was completed, I have gone back to visit several of the classrooms of teachers who first took part in the research. These teachers are still using VNPS and VRG as well as having refined their practice around many of the other nine aforementioned elements. Unlike many other professional development initiatives and interventions I have seen implemented over the years, these really seemed to have had a lasting impact on teacher practice. The reason for this seems to come from two sources. First, teachers talk about how much their students like the 'new' way of doing mathematics. So much so, in fact, that when they go back to using direct instruction, even for brief periods of time, the students object. The second and more intrinsic reason is that they feel more effective as teachers. Their students are exhibiting the traits that they had been striving for but were unable to achieve through nuanced changes to their initial teaching practice.

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